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6170 Section #2
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Plan for today:

- lim sup & lim inf
- Continuous functions
- Open cover examples
- Convex sets

- lim sup & lim inf

$\boxed{\text{Def}}$ The limit superior of x_n is $\limsup_{n\to\infty} x_n := \lim_{m\to\infty} \sup\{x_n : n \geq m\}$.

The limit inferior is $\liminf_{n\to\infty} x_n := \lim_{m\to\infty} \inf\{x_n : n \geq m\}$.

One way to interpret this definition is to define a sequence:
$y_m = \sup\{x_n : n \geq m\}$, the sup of the tail sequence $x_{n \geq m}$.
Then $\limsup_{n\to\infty} x_n = \lim_{m\to\infty} y_m$. So lim sup of x_n is just the limit of the supremum of the tail of the sequence as we move further into the tails.

$\boxed{\text{E.g.}}$ $x_n = \frac{1}{n}$.

What's $\{x_n : n \geq m\}$? $\quad \{\overset{x_m}{\frac{1}{m}}, \overset{x_{m+1}}{\frac{1}{m+1}}, \overset{\cdots}{\cdots}\}$

What's $\sup\{x_n : n \geq m\}$? $\quad x_m = \frac{1}{m}$

What's $\lim_{m\to\infty} \sup\{x_n : n \geq m\}$? $= \lim_{m\to\infty} x_m = \lim_{m\to\infty} \frac{1}{m} = 0$

$\boxed{\text{E.g.}}$ It looks like $\lim x_n$ & $\limsup x_n$ are quite similar, but they're actually different. Take, for example, $x_n = (-1)^n$.

We know x_n doesn't converge. But $y_m := \sup\{x_n : n \geq m\}$ actually behaves nicely: $y_m = 1 \ \forall m$! (If you're not convinced, write out a couple terms of y_m.)

So $\limsup_{n\to\infty} x_n = \lim_{m\to\infty} y_n = 1$.

- ## Continuous functions

Proposition 21. *Let f and g be real-valued functions that are continuous at x_0, and let $k \in \mathbb{R}$. Then the following functions are all continuous at x_0: (i) $|f|$; (ii) kf; (iii) $f+g$; (iv) fg; (v) f/g, if $g(x_0) \neq 0$.*

$f+g$:

f & g are continuous at x_0:

we have $\forall \frac{\varepsilon}{2} > 0, \exists \delta_1$ s.t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2}$

$\delta_2 \quad |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2}$.

Take $\delta = \min\{\delta_1, \delta_2\}$

Relate these ?

WTS: $|x - x_0| < \delta \Rightarrow |f(x) + g(x) - (f(x_0) + g(x_0))| < \varepsilon$

$$|f(x) + g(x) - f(x_0) - g(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So we have $\forall \varepsilon > 0, \exists \delta$ s.t. $|x - x_0| < \delta$

$\Rightarrow |f(x) + g(x) - f(x_0) - g(x_0)| < \varepsilon$.

fg : This one is very similar to the proof for "If $a_n \to a$, $b_n \to b$, then $a_n b_n \to ab$." See last Friday's (9/3) section notes.

f & g are continuous at x_0:

$\forall \varepsilon' > 0, \exists \delta_1$ s.t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon'$

$\delta_2 \quad |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon'$.

Take $\delta = \min\{\delta_1, \delta_2\}$.

WTS: $\forall \varepsilon > 0, \exists \delta$ s.t. $|x - x_0| < \delta \Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$

$$|f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$$
$$\leq |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$$
$$= |f(x)|\underbrace{|g(x) - g(x_0)|}_{<\varepsilon'} + |g(x_0)|\underbrace{|f(x) - f(x_0)|}_{<\varepsilon'}$$

Like last time, we need to find bounds for $|f(x)|$ and $|g(x_0)|$.

How do we know $|f(x)|$ is bounded for x s.t. $|x - x_0| < \delta$?

By continuity of f! We have $|f(x) - f(x_0)| < \varepsilon' \ \forall\, x$ s.t. $|x - x_0| < \delta$.

So $\sup\limits_{|x-x_0|<\delta} |f(x)| = M$ exists. We can choose $M > 0$ s.t. $|g(x_0)| < M$.

(You can also write $|f(x)| < |f(x_0)| + \varepsilon'$, where $|f(x_0)|$ is just a constant)

$$|f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$$

$$\le \underbrace{|f(x)|}_{<M}\underbrace{|g(x) - g(x_0)|}_{<\varepsilon'} + \underbrace{|g(x_0)|}_{<M}\underbrace{|f(x) - f(x_0)|}_{<\varepsilon'}$$

$$< 2M\varepsilon' \equiv \varepsilon$$

So we have $\forall\, \varepsilon > 0$, take $\varepsilon' = \frac{\varepsilon}{2M}$, then $\exists\, \delta$ s.t. $|x - x_0| < \delta$

$\Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$.

$\frac{f}{g}$: idea of proof: Show if g is cont. at x_0. Then $\frac{1}{g}$ is cont. at x_0 ($g(x_0) \neq 0$). Then immediately conclude $f \cdot (\frac{1}{g})$ is cont. at x_0 since both f & $(\frac{1}{g})$ are cont. at x_0.

Or you can prove this directly using δ-ε def.

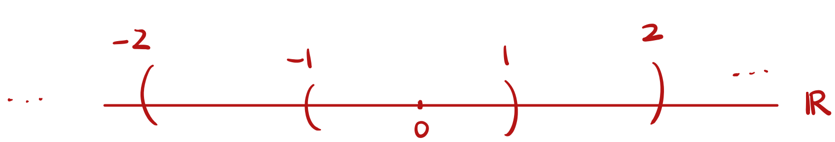
- Open covers & topological compactness.

Def An open cover $\{U_n\}$ for a set $A \subseteq \mathbb{R}^k$ is a collection of open sets U_n whose union contains A: $A \subseteq \bigcup_n U_n$

E.g. Consider $A = \mathbb{R}^1$, the real line.

Then one open cover would be $\{U_n = (-n, n)\}_{n \in \mathbb{N}}$

since $\bigcup_{n=1}^{\infty} U_n = (-1,1) \cup (-2,2) \cup (-3,3) \cup \cdots \supseteq \mathbb{R}$



$\boxed{\text{Def}}$ <u>Topological compactness</u> : A is compact if every open cover of A has a finite subcover.

Q : Is $\mathbb{R}$ compact ? No ! Can't find a finite subcover for $U_n = (-n, n)$

$\boxed{\text{Def}}$ <u>Sequentially compactness</u> : A is sequentially compact if every sequence has a convergent subsequence converging to a pt in A.

Topological compactness $\Leftrightarrow$ sequential compactness.

And in Euclidean space, we have a nice <u>Thm</u> :

$\boxed{\text{Thm}}$ Set $A \subseteq \mathbb{R}^k$ is sequentially compact iff it's closed & bounded

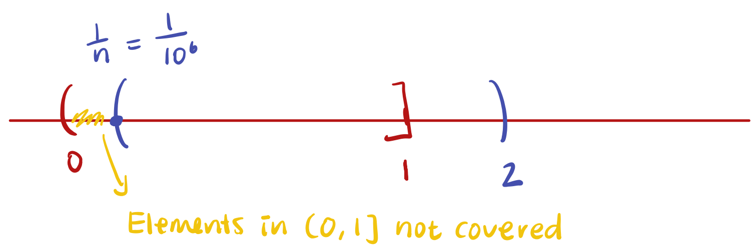
$\boxed{\text{E.g.}}$ $A = [-123, 321]$.

Is $\{U_n = (-n, n)\}_n$ an open cover ? Yes.

Can you find a finite subcover ? $\{U_n\}_{n=1}^{322}$!

$\boxed{\text{E.g.}}$ $A = (0, 1]$. Find an open cover of A that doesn't have a finite subcover.

Take $U_n = (\frac{1}{n}, 2)$, $n = 1, 2, \cdots$. Then $\bigcup_{n=1}^{\infty} U_n = (1,2) \cup (\frac{1}{2}, 2) \cup \cdots$ contains A. But there's no finite subcover!



- Convex sets

$\boxed{\text{Def}}$ The convex hull $CH(X)$ of a set $X \subseteq \mathbb{R}^k$ is the smallest convex set containing X.

$\boxed{\text{Ex.}}$ Prove that the intersection of all convex sets containing X (denoted S) is $CH(X)$.

If we want to show $S = CH(X)$, we need to show $S \subseteq CH(X)$ and $CH(X) \subseteq S$.

Since $CH(X)$ is one of those convex sets that contain X, we have $S \subseteq CH(X)$.

Since S is a convex set containing X, and $CH(X)$ is the smallest convex set containing X by definition, we have $CH(X) \subseteq S$.

$\Rightarrow S = CH(X)$.

Proposition 3. *Let $X \subseteq R^n$ be convex, $\{\alpha_1, \ldots, \alpha_m\}$ a set of $m \geq 1$ real numbers $\in [0,1]$ such that $\sum_{i=1}^m \alpha_i = 1$, and $\{x_1, \ldots, x_m\} \subset X$. Then $\sum_{i=1}^m \alpha_i x_i \in X$.*

Proof of this is by induction:

Base case: $m = 2$

Take $x_1, x_2 \in X$, $\alpha \in [0,1]$. Then $\alpha x_1 + (1-\alpha) x_2 \in X$ since X is convex.

Induction step:

Suppose $\sum_{i=1}^k \alpha_i x_i \in X$ holds for $m = k > 2$. WTS this holds for $m = k+1$.

We have $y = \sum_{i=1}^{k+1} \alpha_i x_i$, $\sum_{i=1}^{k+1} \alpha_i = 1$, and we WTS $y \in X$.

$$= \sum_{i=1}^{k} \alpha_i x_i + \alpha_{k+1} x_{k+1}$$

→ Looks like a convex combo of 2 parts!

$$= (1-\alpha_{k+1}) \boxed{\sum_{i=1}^{k} \frac{\alpha_i}{(1-\alpha_{k+1})} x_i} + \alpha_{k+1} x_{k+1}$$

Note $\sum_{i=1}^{k} \frac{\alpha_i}{1-\alpha_{k+1}} = 1$ since $\sum_{i=1}^{k} \alpha_i + \alpha_{k+1} = 1 \Rightarrow \sum_{i=1}^{k} \alpha_i = 1 - \alpha_{k+1}$

And the induction hypothesis gives $\sum_{i=1}^{k} \frac{\alpha_i}{(1-\alpha_{k+1})} x_i \in X$.

Then h t we end up w/ is a convex combo of 2 elements $\in X$

$\Rightarrow \sum_{i=1}^{k+1} \alpha_i x_i \in X$.