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6170 Section #3
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<u>Quick sketch of what we've covered so far</u>

Module 1
- sup, inf, completeness axiom of $\mathbb{R}$, Archimedean prop.
- sequences, convergence of sequences
    - bounded sequences, monotone seq
    - subsequences
    - Bolzano-Weierstrass Thm
- (lim sup & lim inf (HW2))

Module 2
- Cauchy sequences
- sets: open, closed, compact < topological / sequential
    - Heine-Borel Thm
    - Bolzano-Weierstrass Thm (HW3)
- Continuous functions: sequential, δ-ε def.
    - EVT & IVT

Module 3
- Convex sets
    - convex hull
- Caratheodory's Thm
- Separating hyperplane thm (weak & strong)
- Supporting hyperplane thm
- (quasi-) concave / convex func.

General tips: ① Start w/ the definition!
② Write down what's given & what you need to show.

$\boxed{\text{EVT}}$ $f:[a,b] \to \mathbb{R}$ cont.. Then f is bounded and attains its max on $[a,b]$.

① Show f is bounded on $[a,b]$

Suppose f is not bounded on $[a,b]$. Then $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in [a,b]$ s.t. $|f(x_n)| > n$. Take the seq x_n. Since $[a,b]$ is bounded, x_n has a subseq $x_{n_k} \to x \in [a,b]$ (by B-W, and $[a,b]$ is closed).

Sequential continuity of f gives $x_{n_k} \to x \Rightarrow f(x_{n_k}) \to f(x)$

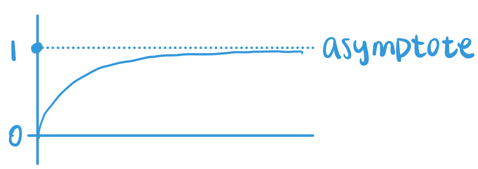
But we have $\forall\, n_k \in \mathbb{N}$, $|f(x_{n_k})| > n_k$

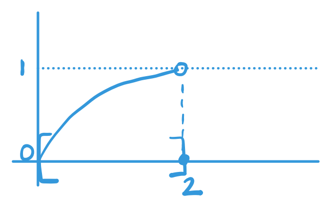
$$\Rightarrow \lim_{n_k \to \infty} |f(x_{n_k})| \geq \lim_{n_k \to \infty} n_k = \infty$$

Contradiction. So f is bounded.

② f attains its max on $[a,b]$

Since we know f is bounded, $M = \sup\{f(x) : x \in [a,b]\}$ exists (by the completeness axiom; also note that M may not be in range(f) for bounded f in general:





We want to show M is in range(f).

Since M is the sup, $\forall\, \varepsilon > 0$, $\exists\, y_n \in [a,b]$ s.t. $f(y_n) > M - \varepsilon$. Take $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. Construct a sequence $y_n \in [a,b]$ s.t. $f(y_n) > M - \frac{1}{n}$. Take limit as $n \to \infty$ on both sides, then $\lim f(y_n) \geq M$. But we also have $f(y_n) \leq M \;\forall\, n$
$\Rightarrow \lim f(y_n) = M$.

Since $[a,b]$ is bounded, B-W says y_n has a convergent subseq $y_{n_k} \to y \in [a,b]$. By sequential continuity of f, $f(y_{n_k}) \to f(y)$.
But $f(y_n)$ & $f(y_{n_k})$ both converge $\Rightarrow f(y) = M$. So M is attained.

(module 1, Ex 24)

$\boxed{\text{Ex.}}$ If f & g are concave on X, then $f+g$ also concave on X.

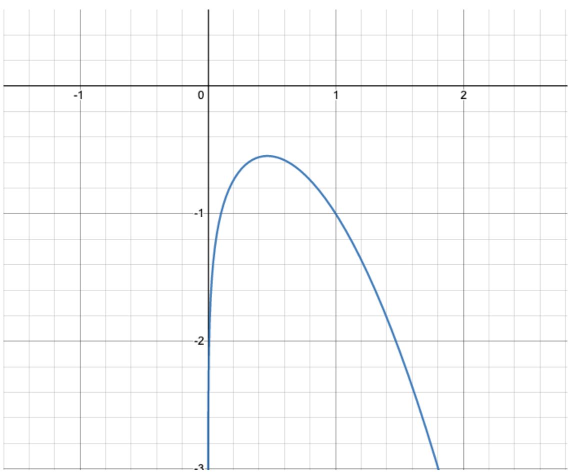
Start w/ def: $\forall\, x_1, x_2 \in X,\ \lambda \in [0,1],\ f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$

$$g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda g(x_1) + (1-\lambda) g(x_2)$$

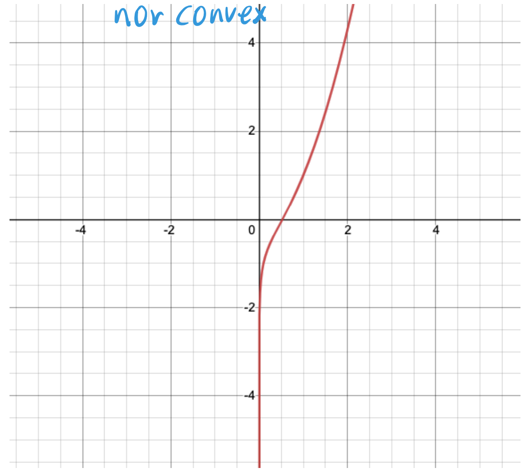
<u>WTS</u>: $f(\lambda x_1 + (1-\lambda)x_2) + g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda(f(x_1) + g(x_1)) + (1-\lambda)(f(x_2) + g(x_2))$

$$[f+g](\lambda x_1 + (1-\lambda)x_2) \geq \lambda [f+g](x_1) + (1-\lambda)[f+g](x_2)$$

$\boxed{\text{E.g.}}$ $\underbrace{h(x)}_{\text{concave}} = \underbrace{\log x}_{\text{concave}} + \underbrace{(-x^2)}_{\text{concave}}$

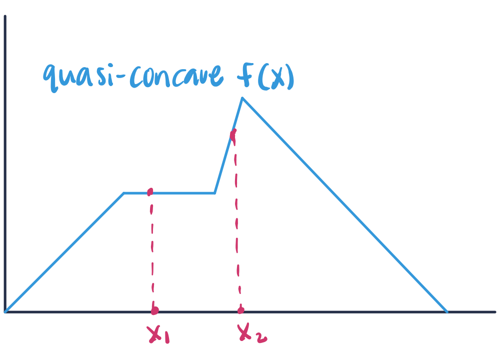


$\boxed{\text{E.g.}}$ $\underbrace{h(x)}_{\text{not concave nor convex}} = \underbrace{\log x}_{\text{concave}} + \underbrace{x^2}_{\text{convex}}$

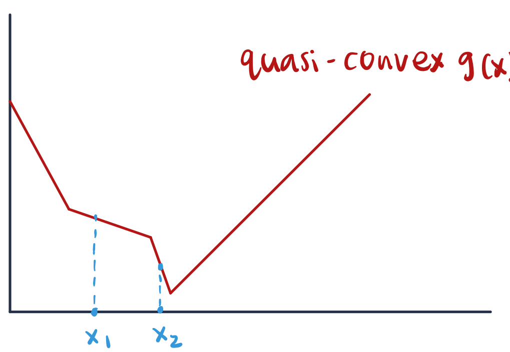


quasi-concave f:
$\forall\, x_1, x_2 \in X,\ \lambda \in [0,1]$,
$f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$

quasi-convex f:
$\forall\, x_1, x_2 \in X,\ \lambda \in [0,1]$
$f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\}$



Note that $f(x)$ is quasi-concave but not concave (fails at x_1 & x_2 and any $\lambda \in (0,1)$).



$g(x)$ is quasi-convex but not convex.

$\boxed{\text{Ex.}}$ If f is convex, then $\forall\, r \in \mathbb{R}$, the set $A \equiv \{x \in X : f(x) \leq r\}$ is convex.

How did we define convex func in class? "f is convex if $-f$ is concave."

This is equivalent to:

f convex : $\forall\, x_1, x_2 \in X$, $\lambda \in [0,1]$, $f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

<u>WTS</u>: $\forall\, x_1, x_2 \in A$, $\forall\, \lambda \in [0,1]$, $\lambda x_1 + (1-\lambda) x_2 \in A$

$\Leftrightarrow$ $\forall\, x_1, x_2$ s.t. $f(x_1) \leq r$, $f(x_2) \leq r$, we have $f(\lambda x_1 + (1-\lambda) x_2) \leq r$

We know from convexity of f:

$$\begin{aligned} f(\lambda x_1 + (1-\lambda) x_2) &\leq \lambda f(x_1) + (1-\lambda) f(x_2) \\ &\leq \lambda r + (1-\lambda) r = r \end{aligned}$$

$\Rightarrow$ $\lambda x_1 + (1-\lambda) x_2 \in A$

$\Rightarrow$ A is convex