1. (5pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

   Let $h = f * g$, i.e., $h(x) = f(x) * g(x)$ for all $x \in \mathbb{R}$.

   If f and g are continuous, prove that h is continuous.

   Note: **Prove it directly using the definition of continuity**. You **may not** simply cite the proposition in the notes that states this result.

f & g are continuous at x_0:

$\forall \varepsilon' > 0, \exists \delta_1$ s.t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon'$

δ_2 $\quad |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon'$.

Take $\delta = \min\{\delta_1, \delta_2\}$.

WTS: $\forall \varepsilon > 0, \exists \delta$ s.t. $|x - x_0| < \delta \Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$

$$|f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$$

$$\leq |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$$

$$= |f(x)|\underbrace{|g(x) - g(x_0)|}_{<\varepsilon'} + |g(x_0)|\underbrace{|f(x) - f(x_0)|}_{<\varepsilon'}$$

$\cancel{\exists}$ f not bounded $\forall x$!

Like last time, we need to find bounds for $|f(x)|$ and $|g(x_0)|$.

How do we know $|f(x)|$ is bounded for x s.t. $|x - x_0| < \delta$?

By continuity of f! We have $|f(x) - f(x_0)| < \varepsilon'$ $\forall x$ s.t. $|x - x_0| < \delta$.

So $\sup_{|x - x_0| < \delta} |f(x)| = M$ exists. We can choose $M > 0$ s.t. $|g(x_0)| < M$.

(You can also write $|f(x)| < |f(x_0)| + \varepsilon'$, where $|f(x_0)|$ is just a cons-tant)

$$|f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)|$$

$$\leq \underbrace{|f(x)|}_{<M}\underbrace{|g(x) - g(x_0)|}_{<\varepsilon'} + \underbrace{|g(x_0)|}_{<M}\underbrace{|f(x) - f(x_0)|}_{<\varepsilon'}$$

$$< 2M\varepsilon' \equiv \varepsilon$$

So we have $\forall \varepsilon > 0$, take $\varepsilon' = \frac{\varepsilon}{2M}$, then $\exists \delta$ s.t. $|x - x_0| < \delta$

$\Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$.

2. (5pts) Let $X \subset \mathbb{R}^k$ be compact and suppose that $f : X \to \mathbb{R}$ is continuous. Prove that $f(X)$ is a compact set in $\mathbb{R}$.

   Note: Recall that $f(X) = \{f(x) | x \in X\}$.

   We use Heine-Borel/Bolzano-Weierstrass, which says a set is compact iff it's closed and bounded.

   You can directly use the extreme value theorem to show that $f(X)$ is bounded because it attains max and min.
   If you want to show this formally, suppose $f(X)$ is not bounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in X$ s.t. $|f(x_n)| > n$. Take the sequence x_n. Since X is compact (hence closed and bounded), x_n has a convergent subsequence $x_{n_k} \to x \in X$ (by Bolzano Weierstrass and closedness of X). By sequential continuity of f, $f(x_{n_k}) \to f(x)$. But for all $n_k \in \mathbb{N}, |f(x_{n_k})| > n_k \implies f(n_{n_k}) \to \infty$. Contradiction. So $f(X)$ is bounded.

   To show $f(X)$ is closed, take any convergent sequence $y_n \in f(X)$. Then $y_n \to y \in \mathbb{R}$. We need to show this limit point $y \in f(X)$.
   For each $y_n \in f(X)$, there exists a corresponding $x_n \in X$. Since X is closed and bounded, x_n has a convergent subsequence $x_{n_k} \to x \in X$. By sequential continuity of f, $f(x_{n_k}) = y_{n_k} \to f(x) \in f(X)$. But $\lim y_{n_k} = \lim y_n \implies y = f(x) \in f(X)$. Hence $f(X)$ is closed.

   Finally, closedness and boundedness of $f(X)$ give us $f(X)$ is compact.

3. (5pts) Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is convex.

   Let $\alpha_1, \dots, \alpha_n \in [0,1]$ be such that $\sum_{i=1}^n \alpha_i = 1$.

   Let $x_1, \dots, x_n \in \mathbb{R}^k$.

   Prove that $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$.

   Note: This is known as Jensen's Inequality.

   Friendly reminder: f is convex if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ for all $\alpha \in [0,1]$ and $x, y \in \mathbb{R}^k$.

   We prove this by induction.
   Base case: $n = 2$ is taken care of by the definition of the convexity of f.

   Induction step: suppose $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$**,** $\sum_{i=1}^n \alpha_i = 1$ **holds for** $n = m > 2$**.**
   Consider $n = m + 1$.

$$
\begin{aligned}
f(\sum_{i=1}^{m+1} \alpha_i x_i) &= f(\sum_{i=1}^{m} \alpha_i x_i + \alpha_{m+1} x_{m+1}) \\
&= f\left((1-\alpha_{m+1}) \sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_{m+1}} x_i + \alpha_{m+1} x_{m+1} \right) \\
&\leq (1-\alpha_{m+1}) f(\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_{m+1}} x_i) + \alpha_{m+1} f(x_{m+1}) \quad (1) \\
&\leq (1-\alpha_{m+1}) \sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_{m+1}} f(x_i) + \alpha_{m+1} f(x_{m+1}) \quad (2) \\
&= \sum_{i=1}^{m+1} \alpha_i f(x_i)
\end{aligned}
$$

   where (1) follows from f is convex, $\mathbb{R}^k$ is convex and module 3 proposition 3; (2) follows from the induction hypothesis for $n = m$.

$\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|.$

4. (5pts) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are convex. Let $h \equiv \max\{f, g\}$, i.e., $h(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$. Prove that h is convex.

   Convexity of f and g gives: for all x and $y \in \mathbb{R}$ and $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$
$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$$

   It follows:

$$\begin{aligned}
&\max\{f(\alpha x + (1-\alpha)y), g(\alpha x + (1-\alpha)y)\} \\
&\leq \max\{\alpha f(x) + (1-\alpha)f(y), \alpha g(x) + (1-\alpha)g(y)\} \\
&\leq \max\left\{\alpha \max\{f(x), g(x)\} + (1-\alpha)\max\{f(y), g(y)\}, \alpha \max\{f(x), g(x)\} + (1-\alpha)\max\{f(y), g(y)\}\right\} \\
&\quad = \alpha \max\{f(x), g(x)\} + (1-\alpha)\max\{f(y), g(y)\}
\end{aligned}$$

   Hence $h = \max\{f, g\}$ is convex.

5. (Extra Credit: 2 pts) Prove that $f(x) = x^{0.5}$ is continuous on $[0, \infty)$.

   Hint 1: When showing continuity at $x_0 \in [0, \infty)$, treat the cases where $x_0 = 0$ and $x_0 > 0$ separately.

   Hint 2: Note that $x - x_0 = (x^{0.5} - x_0^{0.5})(x^{0.5} + x_0^{0.5})$.

   Case 1: $x_0 = 0$
   For all $\epsilon > 0$, $\exists \delta = \epsilon^2$ s.t. $|x - 0| < \delta \implies |x^{0.5} - 0^{0.5}| = x^{0.5} < \delta^{0.5} = \epsilon$.
   Hence $f(x)$ is continuous at $x_0 = 0$.

   Case 2: $x_0 > 0$
   For all $\epsilon > 0$, $\exists \delta = \epsilon \cdot (x_0^{0.5})$ such that
   $|x - x_0| < \delta \implies$

$$
\begin{aligned}
|x^{0.5} - x_0^{0.5}| &= \frac{|x - x_0|}{|x^{0.5} + x_0^{0.5}|} \\
&< \frac{\delta}{x^{0.5} + x_0^{0.5}} = \epsilon \frac{x_0^{0.5}}{x^{0.5} + x_0^{0.5}} \\
&< \epsilon
\end{aligned}
$$

   Hence f is continuous on $[0, \infty)$.

$\boxed{\text{Ex.}}$ If f & g are continuous at x_0. Then $\frac{f}{g}$ is continuous at x_0 ($g(x_0) \neq 0$)

Let's first show "if g is cont. at x_0, then $\frac{1}{g}$ is cont."

g is cont at x_0: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$.

$$\left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| = \left|\frac{g(x_0)}{g(x_0)}\frac{1}{g(x)} - \frac{g(x)}{g(x)}\frac{1}{g(x_0)}\right| = \left|\frac{1}{g(x_0)g(x)}\right| \cdot \underbrace{|g(x_0) - g(x)|}_{<\varepsilon}$$

Neet to find a bound for $\left|\frac{1}{g(x)}\right|$.

Note although $g(x)$ is locally bounded by $(|g(x_0)| - \varepsilon, |g(x_0)| + \varepsilon)$ over $x \in B_\delta(x_0)$, we don't know the magnitude of $||g(x_0)| - \varepsilon|$ and so can't say $\left|\frac{1}{g(x)}\right| < \left|\frac{1}{|g(x_0)| - \varepsilon}\right|$.

A clever way to find a local bound:

① for $\varepsilon = \frac{1}{2}|g(x_0)|$, $\exists \delta_1$ s.t. $|x - x_0| < \delta_1 \Rightarrow |g(x) - g(x_0)| < \frac{1}{2}|g(x_0)|$

and here we can say $\left|\frac{1}{g(x)}\right| < \left|\frac{1}{|g(x_0)| - \varepsilon}\right| = \frac{1}{\frac{1}{2}|g(x_0)|}$

② Take any $\varepsilon > 0$. For this ε, choose $\varepsilon' = \frac{\varepsilon}{2} g(x_0)^2$. Then $\exists \delta_2$ s.t.
$|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2} g(x_0)^2$

Now, for any $\varepsilon > 0$, take $\delta = \min\{\delta_1, \delta_2\}$. Then:

$$\left|\frac{1}{g(x)} - \frac{1}{g(x_0)}\right| = \left|\frac{g(x_0)}{g(x_0)}\frac{1}{g(x)} - \frac{g(x)}{g(x)}\frac{1}{g(x_0)}\right|$$

$$= \underbrace{\left|\frac{1}{g(x_0)g(x)}\right|}_{\text{by ①}} \cdot \underbrace{|g(x_0) - g(x)|}_{\text{by ②}}$$

$$< \frac{1}{\frac{1}{2}|g(x_0)|^2} \cdot \frac{\varepsilon}{2} g(x_0)^2 = \varepsilon.$$

Then, since f & $\frac{1}{g}$ both cont. at x_0, so is $f \cdot \frac{1}{g} = \frac{f}{g}$.

$\boxed{\text{Lemma}}$ $x_n \to x$ iff $\forall$ subsequence x_{n_k}, $\exists$ sub-subsequence $x_{n_{k_l}} \to x$.

<u>"If": $\forall$ subseq x_{n_k}, $\exists\, x_{n_{k_l}} \to x \Rightarrow x_n \to x$</u>

Suppose $x_n \not\to x$. Then $\exists\, \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists\, n > N$ s.t. $|x_n - x| \geq \varepsilon$.

$\exists\, n_1 > 1$ s.t. $|x_{n_1} - x| \geqslant \varepsilon$

$\exists\, n_2 > \max\{2, n_1\}$ s.t. $|x_{n_2} - x| \geqslant \varepsilon$

$\vdots$

Construct a subseqence x_{n_k} by doing so. Then $\forall$ sub-subsequence $x_{n_{k_l}}$, we have $\forall N \in \mathbb{N}$, $n_{k_l} > N$ but $|x_{n_{k_l}} - x| \geqslant \varepsilon \Rightarrow x_{n_k}$ doesn't have a convergent sub-subsequence. Contradiction.

"Only if": $x_n \to x \Rightarrow \forall x_{n_k}\ \exists\, x_{n_{k_l}} \to x$.

Follows from: "If a sequence converges, then every subsequence converges to the same limit." (Module 1, Ex. 24)

$x_n \to x \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |x_n - x| < \varepsilon$

Let x_{n_k} be any subseq. Then $n_k > N \Rightarrow |x_{n_k} - x| < \varepsilon$.