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6170 Section #5
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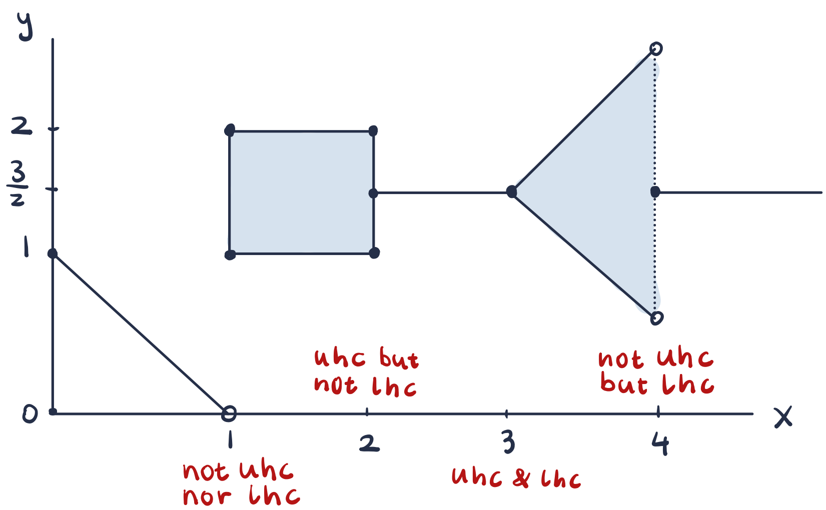
$\boxed{\text{Def}}$ A correspondence $\phi: X \rightrightarrows Y$ is

uhc if $\forall x_n \in X$ s.t. $x_n \to x \in X$ and $\forall y_n \in \phi(x_n)$ s.t. $y_n \to y \in Y$, we have $y \in \phi(x)$

lhc if $\forall x_n \in X$ s.t. $x_n \to x \in X$ and $\forall y \in \phi(x)$, $\exists y_n \in \phi(x_n)$ s.t. $y_n \to y$.

$\boxed{\text{E.g.}}$ Consider $\phi: \mathbb{R}_+ \rightrightarrows \mathbb{R}_{++}$

$$\phi(x) = \begin{cases} \{1-x\}, & x \in [0,1) \\ [1,2], & x \in [1,2] \\ \{\frac{3}{2}\}, & x \in (2,3] \cup [4,\infty) \\ \{y : -x + \frac{9}{2} \le y \le x - \frac{3}{2}\}, & x \in (3,4) \end{cases}$$



At $x=1$, take $x_n = 1 - \frac{1}{n}$,

- $y_n = \frac{1}{n} \in \phi(x_n)$, $y_n \to y = 0 \notin \phi(1) \Rightarrow$ not uhc
- Take any $y \in \phi(1)$. Say $y=1$. Does $\exists$ seq $y_n \in \phi(x_n)$ that converges to $y=1$? No because $\forall y_n \in \phi(x_n)$, $y_n \to 0$. $\Rightarrow$ not lhc.

At $x=2$, take any $x_n \to x$ and $y_n \in \phi(x_n)$ s.t. $y_n \to y \in \mathbb{R}_{++}$. For any $\varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow |x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$. Pick $\varepsilon = 1$. Then $x_n \in (1,3)$ & $y_n \in \underbrace{[1,2] \cup \{\frac{3}{2}\}}_{\text{union of finitely many closed sets is closed.}}$ $\forall n > N \Rightarrow y \in [1,2] \cup \{\frac{3}{2}\} = [1,2] = \phi(2)$

The idea here is that the set of image at some ε-nbh of $x=2$ is closed. So any convergent seq in that set converges to a pt in that set.

Try to formally prove the rest as an exercise.

$\boxed{\text{Thm}}$ Let $f: X \times \Theta \to \mathbb{R}$ be a func, $\phi: \Theta \rightrightarrows X$ a correspondence.

Consider $\max_{z \in \phi(\theta)} f(z, \theta)$.

"policy func" if single-valued

Let $\sigma: \Theta \rightrightarrows X$ defined as $\sigma(\theta) \equiv \arg\max_{z \in \phi(\theta)} f(z,\theta)$, and $f^*: \Theta \to \mathbb{R}$

"value func"

be defined as $f^*(\theta) \equiv \sup\{f(z,\theta) : z \in \phi(\theta)\}$.

If we assume:

① X is closed.

② f is cont. in (z, θ)

③ $\phi: \Theta \rightrightarrows X$ is cont., nonempty-valued and locally bounded.

hence both uhc & lhc.

Then we have:

① $\sigma: \Theta \rightrightarrows X$ is a nonempty-valued, uhc and locally bounded corresp.

② $f^*: \Theta \to \mathbb{R}$ is a cont. func.

Let's walk through the proof for continuity of f^* & uhc of σ again.

To show f^* is cont., WTS: $\forall\ \theta_n \to \theta \in \Theta$, $\lim f^*(\theta_n) = f^*(\theta)$.

First we show $f^*(\theta) \geq \lim f^*(\theta_n)$ by using ϕ is locally bounded & uhc.

Take any seq $\theta_n \to \theta \in \Theta$ and $z_n \in \sigma(\theta_n) \subseteq \phi(\theta_n)$ (σ is nonempty valued)

To use uhc of ϕ, we need a convergent seq in $\phi(\theta_n)$.

How do we find such seq? use local boundedness of ϕ & B-W!

ϕ is locally bounded $\Rightarrow \exists\ \varepsilon > 0$ and bounded set $B \subset X$ s.t. $\|\theta' - \theta\| < \varepsilon$

$\Rightarrow \phi(\theta') \subseteq B \Rightarrow \sigma(\theta') \subseteq \phi(\theta') \subseteq B$. So for this ε, $\exists\ N$ s.t. $n > N$

$\Rightarrow \|\theta_n - \theta\| < \varepsilon \Rightarrow z_n \in \sigma(\theta_n) \subseteq B \Rightarrow z_n$ is bounded.

B-W tells us $\exists$ subseq $z_{n_k} \in \sigma(\theta_{n_k}) \subseteq \phi(\theta_{n_k})$ s.t. $z_{n_k} \to z \in X$.

Let's not skip notation here (:

(X is closed)

But ϕ uhc $\Rightarrow z_{n_k} \to z \in \phi(\theta)$ Note θ_{n_k} is a subseq of θ_n and $\to \theta$.

So $f^*(\theta) \geq f(z,\theta) = \lim f(z_{n_k}, \theta_{n_k}) = \lim f^*(\theta_{n_k})$

by def of f^* and $z \in \phi(\theta)$

by cont. of f

by $z_{n_k} \in \sigma(\theta_{n_k})$ and $f^*(\theta) = f(z,\theta)$ s.t. $z \in \sigma(\theta)$

Next we show $f^*(\theta) \leq \lim f^*(\theta_n)$ by using lhc of Φ.

Take any $z' \in \sigma(\theta) \subseteq \phi(\theta)$. By lhc of ϕ, $\forall \theta_n \to \theta \in \Theta$, $\exists z_n' \in \phi(\theta_n)$ s.t. $z_n' \to z$. Note that z_n' is in $\Phi(\theta_n)$, not necessarily $\sigma(\theta_n)$.

$\Rightarrow f^*(\theta_n) \geq f(z_n', \theta_n) \quad \forall n.$

$\Rightarrow \lim f^*(\theta_n) \geq \lim f(z_n', \theta_n) = f(z', \theta) = f^*(\theta)$

by cont of f

by $z' \in \sigma(\theta)$

$\Rightarrow \lim f^*(\theta_n) = f^*(\theta)$

$\Rightarrow f^*$ is cont.

Now let's show σ is uhc.

$\forall \theta_n \to \theta \in \Theta$ and $z_n \in \sigma(\theta_n)$ s.t. $z_n \to z \in X$. WTS $z \in \sigma(\theta)$.

But $\lim f(z_n, \theta_n) = \lim f^*(\theta_n) = \boxed{f^*(\theta) = f(z,\theta)} \Rightarrow \boxed{z \in \sigma(\theta)}$

by def of f^* & $z_n \in \sigma(\theta_n)$

by cont. of f^*

by cont of f

$\Rightarrow \sigma$ is uhc.

Question: Is σ lhc? Not necessarily!

Take $\Theta = X = [0,1]$ (closed). $f: X \times \Theta \rightrightarrows \mathbb{R}$ to be $f(x,\theta) = x\theta$ (cont.)

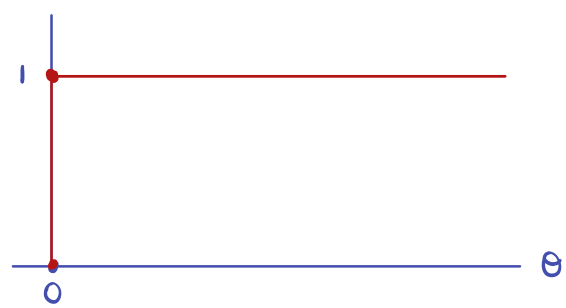
Suppose $\phi: \Theta \rightrightarrows X$ is $\phi(\theta) = [0,1]$ $\forall \theta$ (constant corresp. and hence both uhc & lhc ; good exercise to show). Obvious to see Φ is nonempty & locally bounded.

So the thm of maximum applies.

For any θ, $\sigma(\theta) = \arg\max_{x \in \Phi(\theta)} x\theta$

$$= \arg\max_{x \in [0,1]} x\theta$$

$$= \begin{cases} [0,1] & \text{if } \theta = 0 \\ 1 & \text{if } \theta > 0. \end{cases}$$



Not lhc. Take $\theta_n = \frac{1}{n}$. $\theta_n \to \theta = 0$ and $y = 0 \in \sigma(0)$. $\forall\, y_n \in \sigma(\theta_n)$, $y_n = 1$ (since $\theta_n > 0 \;\forall\, n$). So no $y_n \in \sigma(\theta_n)$ converges to $y = 0$.