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6170 Section #6
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Clarification on HW6, Ex. 5.

Proposition 15. [*A singleton-valued correspondence that is locally bounded is upper semi-continuous if and only if it is lower semi-continuous*] *if and only if* [*the function is continuous.*]

You need to show if ϕ is singleton-valued & locally bounded, then "uhc is equivalent to lhc" iff "ϕ is associated w/ a cont. func."

Note that Prop 13. (which you'll prove in Ex. 4) already tells you if "ϕ is singleton-valued" then "lhc $\Leftrightarrow$ association w/ a cont f."

Proposition 13. *A singleton-valued correspondence $\phi : X \rightrightarrows Y$ is lower semi-continuous if and only if it is associated with a continuous function f. If it is associated with a continuous f, then it is upper semi-continuous.*

What's left is to show "uhc $\Leftarrow$ association w/ a cont f." using the additional assumption that ϕ is locally bounded.

What we didn't finish last time ...

Thm Let $f: X \times \Theta \to \mathbb{R}$ be a func, $\phi : \Theta \rightrightarrows X$ a correspondence.

Consider $\max_{z \in \phi(\theta)} f(z, \theta)$.

Let $\sigma : \Theta \rightrightarrows X$ defined as $\sigma(\theta) \equiv \arg\max_{z \in \phi(\theta)} f(z,\theta)$ ("policy func" if single-valued), and $f^* : \Theta \to \mathbb{R}$ be defined as $f^*(\theta) \equiv \sup\{f(z,\theta) : z \in \phi(\theta)\}$ ("value func").

<u>If we assume</u>:

① X is closed.
② f is cont. in (z, θ)
③ $\phi : \Theta \rightrightarrows X$ is cont., nonempty-valued and locally bounded.
(cont. → hence both uhc & lhc.)

Then we have :

① $\sigma : \Theta \rightrightarrows X$ is a nonempty-valued, uhc and locally bounded corresp.
② $f^* : \Theta \to \mathbb{R}$ is a cont. func.

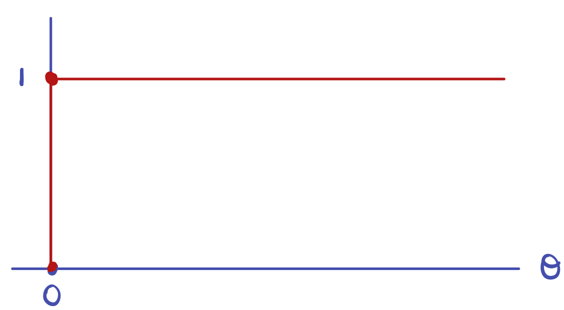
Question : Is σ lhc ? Not necessarily !

Take $\Theta = X = [0,1]$ (closed). $f : X \times \Theta \to \mathbb{R}$ to be $f(x,\theta) = x\theta$ (cont.)
Suppose $\phi : \Theta \rightrightarrows X$ is $\phi(\theta) = [0,1]$ $\forall \theta$ (constant corresp. and hence both uhc & lhc ; good exercise to show). Obvious to see ϕ is nonempty & locally bounded.

So the thm of maximum applies.

For any θ, $\sigma(\theta) = \arg\max_{x \in \phi(\theta)} x\theta$

$$= \arg\max_{x\in[0,1]} x\theta = \begin{cases} [0,1] & \text{if } \theta = 0 \\ 1 & \text{if } \theta > 0. \end{cases}$$



Not lhc. Take $\theta_n = \frac{1}{n}$. $\theta_n \to \theta = 0$ and $y = 0 \in \sigma(0)$. $\forall\, y_n \in \sigma(\theta_n)$, $y_n = 1$ (since $\theta_n > 0$ $\forall n$). So no $y_n \in \sigma(\theta_n)$ converges to $y = 0$.

Prop Let $f : X \times \Theta \to \mathbb{R}$ & $\phi : \Theta \rightrightarrows X$ satisfy the assumptions in Berge's Thm. If we also assume f is quasi-concave in X & ϕ is convex-valued, then the solution correspondence $\sigma : \Theta \rightrightarrows X$ is convex valued. If f is strictly quasi-concave then σ is singleton-valued.

Proof : Fix any $\theta \in \Theta$, $f(x,\theta)$ is quasi-concave on X
$\iff \{x \in X : f(x,\theta) \geq r\}$ is convex $\forall r \in \mathbb{R}$ (by module 3, Ex. 8)

In particular, $A := \{x \in X : f(x,\theta) \geq \underbrace{\max_{z \in \phi(\theta)} f(z,\theta)}_{\in \mathbb{R}}\}$ is convex.

Since $\phi(\theta) = \{x \in X : x \in \phi(\theta)\}$ is convex, $\phi(\theta) \cap A$ is convex.

But $\phi(\theta) \cap A = \{x \in \phi(\theta) : f(x,\theta) = \max_{z \in \phi(\theta)} f(z,\theta)\} = \sigma(\theta)$.

Hence σ is convex-valued.
If in addition, f is strictly quasiconcave, but $\exists\, x_1, x_2 \in \sigma(\theta)$ s.t. $x_1 \neq x_2$. Then take any $\alpha \in [0,1]$. $\alpha x_1 + (1-\alpha)x_2 \in \sigma(\theta)$.
But $f(\alpha x_1 + (1-\alpha)x_2, \theta) > \min\{f(x_1,\theta), f(x_2,\theta)\} = f^*(\theta)$. Contradiction

Ex. If f & g are differentiable at x & $g(x) \neq 0$. Then $\frac{f}{g}$ is differentiable at x and $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$$\frac{f}{g}(t) - \frac{f}{g}(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(t)g(x) - g(t)f(x)}{g(t)g(x)}$$

$$= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - g(t)f(x)}{g(t)g(x)}$$

$$= \frac{g(x)[f(t) - f(x)] - f(x)[g(t) - g(x)]}{g(t)g(x)}$$

$$\lim_{t \to x} \frac{\frac{f}{g}(t) - \frac{f}{g}(x)}{t - x} = \lim_{t \to x} \frac{1}{g(t)g(x)}\left[g(x)\frac{f(t) - f(x)}{t - x} - f(x)\frac{g(t) - g(x)}{t - x}\right]$$

$$= \frac{1}{g(x)^2}\left[g(x)f'(x) - f(x)g'(x)\right]$$

by "if $g(x)$ diff. at x, then $g(x)$ cont. at x"
and "if $g(x)$ cont at x and $g(x) \neq 0$, then $\frac{1}{g}(x)$ cont. at x."