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Constrained Optimization w/ Inequality Constraint

Let $f: \mathbb{R}^n \to \mathbb{R}$ and $g(x): \mathbb{R}^n \to \mathbb{R}^m$. Consider $\max_x f(x)$ s.t. $g(x) \leq y$ $(y \in \mathbb{R}^m)$ (✯)
(i.e., we have m constraints: $g_1(x) \leq y_1, \cdots, g_m(x) \leq y_m$)

$\boxed{\text{Constraint qualification}}$ holds if for any feasible $x \in \mathbb{R}^n$, the set of vectors $\{Dg_i(x) : g_i(x) = y_i\}$ is linearly independent.

Exercise from class: T/F: if $m > n$, then CQ doesn't hold $\forall x \in \mathbb{R}^n$
Take $m=2, n=1$: $f(x) = \log x$, $g_1(x) = -x \leq 0$, $g_2(x) = x^2 \leq 4$
Obviously the CQ holds for $x = 2$,
in which case $\{Dg_i(x) : g_i(x) = y_i\} = \{Dg_2(x)\} = \{[2x]|_{x=2}\} = \{[4]\}$
set of linearly independent vector in $\mathbb{R}$.

$\boxed{\text{FOC}}$ If at $x \in \mathbb{R}^n$ and some $\lambda \in \mathbb{R}^m_+$, $\underbrace{\nabla f(x)}_{1\times n} = \underbrace{\lambda'}_{1\times m} \underbrace{Dg(x)}_{m \times n}$.

$$\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1} \cdots \frac{\partial f(x)}{\partial x_n}\right] \qquad Dg(x) = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m(x)}{\partial x_1} & \cdots & \frac{\partial g_m(x)}{\partial x_n} \end{bmatrix}$$

Then we say the FOC of the constrained optimization prob ✯ holds at x w/ λ.

$\boxed{\text{CS}}$ holds at $x \in \mathbb{R}^n$ w/ $\lambda \in \mathbb{R}^m_+$ if $\lambda_i [y_i - g_i(x)] = 0$ $\forall i \in \{1, \cdots, m\}$

$\boxed{\text{KKT}}$ Necessary Conditions for Optimality

If $x^* \in \mathbb{R}^n$ solves ✯ and CQ holds at x^*. Then $\exists \lambda^* \in \mathbb{R}^m_+$ s.t. FOC & CS hold at x^* w/ λ^*.

Sufficient conditions

If f is concave and each g_i is quasiconvex for $i \in \{1, \cdots, m\}$.
If a feasible $x^* \in \mathbb{R}^n$ & some $\lambda^* \in \mathbb{R}^m_+$ satisfy FOC & CS. Then x^* solves ✯.

Constrained Optimization w/ equality constraint

Let $f: \mathbb{R}^n \to \mathbb{R}$ and $g(x): \mathbb{R}^n \to \mathbb{R}^m$. Consider $\max_x f(x)$ s.t. $g(x) = y$ $(y \in \mathbb{R}^m)$ (☆☆)
(i.e., we have m constraints: $g_1(x) = y_1, \cdots, g_m(x) = y_m$)

analogue to KKT

$\boxed{\text{Lagrange's Thm}}$ Suppose that $x^* \in \mathbb{R}^n$ solves ☆☆. If the set of vectors $\{Dg_i(x^*) \mid i = 1, \cdots, m\}$ is linearly independent. Then $\exists\, \lambda^* \in \mathbb{R}^m$ s.t. the FOC holds at x^* w/ λ^*.

Q: What's the difference between CQ in KKT & the Lagrange cond.?

E.g., $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ & 1 equality constraint: $g(x) = 4x = 4$

$$\Leftrightarrow \quad \max_x \; x \quad \text{s.t. } 4x = 4 \quad \Rightarrow \quad x^* = 1$$

$\{Dg_i(x^*) \mid i = 1, \cdots, m\}$ in this example? $\{4\}$! which is linearly independent in $\mathbb{R}$.

Rewrite this equality constrained problem in inequalities.

$$\Leftrightarrow \quad \max_x \; x \quad \text{s.t. } \underbrace{4x}_{g_1(x)} \le 4 \quad \& \quad \underbrace{-4x}_{g_2(x)} \le -4$$

Do the 2 inequalities bind at $x^* = 1$? Yes!

What's $\{Dg_i(x^*) \mid g_i(x^*) = y_i\}$, the set for CQ in KKT? $\underbrace{\{4, -4\}}_{\text{Linearly dependent!}}$

In general, if we have m equality constraints, we'll have 2m binding inequality constraints in the KKT framework (2 ineq. for each 1 eq. Also note that the gradients of these 2 ineq. will be the negative of each other, hence linearly dependent.)

What the Lagrange cond. does is to get around CQ by eliminating half of the 2m ineq. gradients that are 100% going to be lin. dep. if included, leaving the m ones that will likely be lin. indep.

Example (adapted from June 2001 micro Q part IV)

Consider two individuals, 1 & 2, consuming 2 goods, x & y, w/ the following utility functions:

$$f^1(x_1,y_1) = x_1 - \gamma y_2, \text{ where } \gamma \in [0,1)$$

negative externality from person 2's consumption of good y!

$$f^2(x_2,y_2) = (x_2 y_2)^{\frac{1}{2}}$$ (this satisfies Inada cond.)

Each individual is endowed w/ 1 unit of each good. Let the price of good x be 1 and the price of y be $P>0$.

Defn: a competitive equilibrium allocation $((x_1^*, y_1^*), (x_2^*, y_2^*))$ and price P^* for this economy is the solution to the 2 constrained optimization Prob:

$$\left.\begin{array}{ll}(1) & \max_{x_1,y_1} f^1(x_1,y_1) \text{ s.t. } \underbrace{x_1 + Py_1}_{g^1(x_1,y_1)} \leq 1+P \\ (2) & \max_{x_2,y_2} f^2(x_2,y_2) \text{ s.t. } \underbrace{x_2 + Py_2}_{g^2(x_2,y_2)} \leq 1+P\end{array}\right\} (n=2, m=1)$$

and market clearing condition holds given P^*: $x_1^* + x_2^* = 2$
$y_1^* + y_2^* = 2$

(I omitted nonnegativity constraints for simplicity; they will not alter the results in this example)

Q1 Check the objective functions are concave, and the constraints $g^1(x_1,y_1) = x_1 + py_1$ & $g^2(x_2,y_2) = x_2 + Py_2$ are quasiconvex.

$f^1(x_1,y_1) = x_1 - \gamma y_2$ is linear ✓

$f^2(x_2,y_2) = (x_2 y_2)^{\frac{1}{2}}$ check the hessian $H(x_2,y_2)$ is NSD $\forall (x_2,y_2) \in \underline{\mathbb{R}^2_{++}}$ why? by checking all principal minors of $H(x_2,y_2)$ have sign $(-1)^k$ or $=0$.

$$\nabla f^2(x_2,y_2) = \left[\tfrac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}} \quad \tfrac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}}\right]$$

$$H_{f^2}(x_2,y_2) = \begin{bmatrix} \overbrace{-\tfrac{1}{4} x_2^{-\frac{3}{2}} y_2^{\frac{1}{2}}}^{a} & \overbrace{\tfrac{1}{4} x_2^{-\frac{1}{2}} y_2^{-\frac{1}{2}}}^{b} \\ \underbrace{\tfrac{1}{4} x_2^{-\frac{1}{2}} y_2^{-\frac{1}{2}}}_{c} & \underbrace{-\tfrac{1}{4} x_2^{\frac{1}{2}} y_2^{-\frac{3}{2}}}_{d} \end{bmatrix}$$

1st order: $a < 0$, $d < 0$

2nd order: $ad - bc = \frac{1}{16} x_2^{-1} y_2^{-1} - \frac{1}{16} x_2^{-1} y_2^{-1} = 0$

$\Rightarrow f^2$ is concave

Both g^1 & g^2 are linear so they're convex & therefore quasiconvex.

Sufficient conditions ✓

If f is concave and each g_i is quasiconvex for $i \in \{1, \cdots, m\}$.

If a feasible $x^* \in \mathbb{R}^n$ & some $\lambda^* \in \mathbb{R}^m_+$ satisfy FOC & CS. Then x^* solves ✱.

<u>Q2</u> Solve (1) & (2) by finding feasible (x_j^*, y_j^*) & $\lambda_j^* \geq 0$ for individual $j = 1, 2$ satisfying $\nabla f^j(x_j^*, y_j^*) = \lambda_j^* D g^j(x_j^*, y_j^*)$ & $\lambda_j^* (1+P - g^j(x_j^*, y_j^*)) = 0$ (Or doing something else ⋯)

For individual 1: $\max\limits_{x_1, y_1} x_1 - \gamma y_2$ s.t. $x_1 + P y_1 \leq 1 + P$

FOC gives: $[1 \;\; 0] = \lambda_1 [1 \;\; P]$. We imediately see that $\nexists\, \lambda_1 \geq 0$ that satisfy FOC. Note that this is not in conflict w/ the sufficient cond. Here we actually have a corner solution.
Note that 1's utility does not depend on consuming good y.
So $x_1^* = 1+P$ and $y_1^* = 0$.

(Alternatively, you can solve (1) by noting that the constraint $x_1 + P y_1 \leq 1+P$ is effectively the same as $x_1 \leq 1+P$ since y_1 doesn't matter. Then we have $n=1$ & $m=1$.
FOC gives $1 = \lambda_1^*$ and CS gives $\lambda_1^*(1+P-x_1) = 0 \Rightarrow x_1^* = 1+P$)

For individual 2: $\max\limits_{x_2, y_2} (x_2 y_2)^{\frac{1}{2}}$ s.t. $x_2 + P y_2 \leq 1 + P$

FOC: $[\frac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}} \;\; \frac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}}] = \lambda_2 [1 \;\; P]$

$$\Leftrightarrow \begin{cases} \frac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}} = \lambda_2 \\ \frac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}} = P \lambda_2 \end{cases}$$

Note the BC will bind since f^2 is strictly increasing in both x_2, y_2.
So $x_2 + Py_2 = 1+P$ and CS: $\lambda_2 [1+P-(x_2+Py_2)] = 0$ holds $\forall \lambda_2 \geq 0$
By Inada cond., we know $x_2 > 0$ & $y_2 > 0 \Rightarrow \lambda_2 > 0$ by FOC.

$$\frac{\frac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}}}{\frac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}}} = \frac{\lambda_2}{P\lambda_2} \Leftrightarrow \frac{y_2}{x_2} = \frac{1}{P}$$

BC: $x_2 + Py_2 = 1+P \Rightarrow y_2 = \frac{(1+P) - x_2}{P}$

$$\Rightarrow \frac{(1+P) - x_2}{Px_2} = \frac{1}{P} \Rightarrow \left.\begin{aligned} x_2^* &= \frac{1+P}{2} \\ y_2^* &= \frac{1+P}{2P} \end{aligned}\right\} \lambda_2^* = \frac{1}{2}\sqrt{\frac{1}{P}}$$

How to solve for P^*? Use market clearing!

$$x_1^* + x_2^* = 2 = 1 + P + \frac{1+P}{2} \Rightarrow P^* = \frac{1}{3}$$

So in CE, $\boxed{\begin{aligned} x_1^* &= \tfrac{4}{3}, \quad y_1^* = 0 \\ x_2^* &= \tfrac{2}{3}, \quad y_2^* = 2 \end{aligned}}$

Q3 For what values of γ is the above allocation PO?

Defn $(x_1^*, x_2^*, y_1^*, y_2^*)$ is PO if $\exists \bar{u}$ s.t. this allocation solves

$$(3) \quad \max_{x_1, x_2, y_1, y_2} \; x_1 - \gamma y_2 \quad \text{s.t.} \quad (x_2 y_2)^{\frac{1}{2}} \geq \bar{u} \quad (\text{or } -(x_2y_2)^{\frac{1}{2}} \leq -\bar{u})$$
$$x_1 + x_2 \leq 2$$
$$y_1 + y_2 \leq 2$$

For now let's take $\bar{u}$ as given. Since we know BC / feasibility constraint will bind, the constrained problem above is the same as:

$$\max_{x_2, y_2} \; (2 - x_2) - \gamma y_2 \quad \text{s.t.} \quad -(x_2y_2)^{\frac{1}{2}} \leq -\bar{u}$$
$$x_2 \leq 2 \quad \text{(by substituting out } x_1\text{)}$$
$$y_2 \leq 2 \quad \text{(since } y_1 \text{ is not in the obj. func.)}$$

Note that now we're assuming $(x_1^*, x_2^*, y_1^*, y_2^*)$ from Q2 solves (3) and want to back out γ that makes ⤴ so.

What do we want to use then? The necessary condition!

$\boxed{\text{KKT}}$ Necessary conditions for optimality

If $x^* \in \mathbb{R}^n$ solves ✮ and CQ holds at x^*. Then $\exists\ \lambda^* \in \mathbb{R}^m_+$ s.t. FOC & CS hold at x^* w/ λ^*.

Does CQ hold at $\boxed{\begin{array}{ll} x_1^* = \frac{4}{3}, & y_1^* = 0 \\ x_2^* = \frac{2}{3}, & y_2^* = 2 \end{array}}$?

call these values M^*

Binding constraints :

① $(x_2^* y_2^*)^{\frac{1}{2}} = \bar{u} \Rightarrow \left[\frac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}} \quad \frac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}}\right]\Big|_{\substack{x_2 = x_2^* = \frac{2}{3} \\ y_2 = y_2^* = 2}}$

$$= [0.87 \quad 0.29]$$

② $y_2^* = 2 \Rightarrow [0 \quad 1]\big|_{y_2 = y_2^* = 2} = [0 \quad 1]$

Apparently $[0.87 \quad 0.29]$ & $[0 \quad 1]$ are linearly indep.. So we can use KKT necessary cond.!

Want to find nonnegative $\lambda^* = [\lambda_1^* \quad \lambda_2^* \quad \lambda_3^*]$ s.t. the FOC & CS hold at M^*

CS : $\lambda_1(-\bar{u} + (x_2^* y_2^*)^{\frac{1}{2}}) = 0 \Rightarrow \lambda_1^* \geq 0$ since $\bar{u} = (x_2^* y_2^*)^{\frac{1}{2}}$

$\lambda_2(2 - x_2^*) = 0 \Rightarrow \lambda_2^* = 0$ since $2 - x_2^* \neq 0$

$\lambda_3(2 - y_2^*) = 0 \Rightarrow \lambda_3^* \geq 0$ since $2 = y_2^*$

FOC :

$$[-1 \quad -r] = \lambda^T \begin{bmatrix} -\frac{1}{2} x_2^{-\frac{1}{2}} y_2^{\frac{1}{2}} & -\frac{1}{2} x_2^{\frac{1}{2}} y_2^{-\frac{1}{2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\Bigg|_{M^*} = \lambda^T \begin{bmatrix} -0.87 & -0.29 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} -1 = -0.87\lambda_1^* + \underbrace{\lambda_2^*}_{\text{"0 from CS."}} = 1.32\lambda_1^* \Rightarrow \lambda_1^* = 1.15 \\ -r = -0.29\lambda_1^* + \lambda_3^* \Rightarrow \lambda_3^* = 0.29(1.15) - r \\ \qquad\qquad\qquad\qquad\qquad\quad = 0.33 - r \geq 0 \end{cases}$$

$\Rightarrow r \leq 0.33$

Note that you don't need to solve Q3 numerically! I'm doing this

Just to make the example more concrete.

In fact, if you do it algebraically you'll get exactly $\gamma \leq p^* = \frac{1}{3}$!

The intuition is, individual 1 doesn't want individual 2 to consume good y (bc of the $-\gamma y_2$ term in f': negative externality).

But if the benefit of selling good y (p^*) is greater the negative impact/cost of doing so (γ), indiv. 1 will be willing to sell and the CE in Q2 holds.